Average Causal Effect Estimation in DAGs with Hidden Variables: Extensions of Back-Door and Front-Door Criteria

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Motivation

- Identification theory for causal effects in ADMGs is well developed but methods for estimation are still limited.
- Existing estimation approaches suffers from computational challenges, under-explored asymptotic property, and may produce estimates outside of the target parameter space.
- A more flexible and robust estimation approach is needed.

Back-Door & Front-Door Models

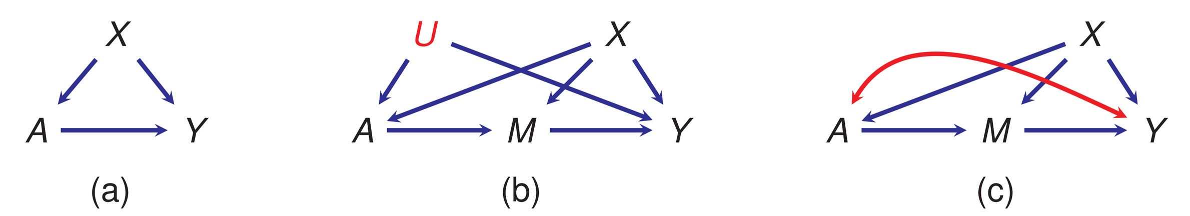


Figure 1: (a) Back-door model; (b) Front-door model (DAG); (c) Front-door model (ADMG).

Target parameter. $\psi_{a_0} := \mathbb{E}[Y^{a_0}],\ a_0 \in \{0,1\}$

Identification functional.

$$\psi_{a_0}(P) = \int y\, p(y \mid a_0, x)\, p(x)\, dy\, dx. \qquad \text{(back-door)}$$

$$\psi_{a_0}(P) = \iint \sum_{a=0}^{1} \mathbb{E}(Y \mid m, a, x)\, p(a \mid x)\, p(m \mid a_0, x)\, p(x)\, dm\, dx. \qquad \text{(front-door)}$$

Efficient influence functional (EIF).

$$\Phi(P) = \frac{\mathbb{I}(A = a_0)}{p(a_0 \mid X)}\left(Y - \mathbb{E}[Y \mid a_0, X]\right) + \mathbb{E}[Y \mid a_0, X] - \psi_{a_0}(P). \qquad \text{(back-door)}$$

$$\Phi(P) = \frac{p(M \mid a_0, X)}{p(M \mid A, X)}\{Y - \mathbb{E}[Y \mid M, A, X]\} + \frac{\mathbb{I}(A = a_0)}{p(a_0 \mid X)}\{\xi(M, X) - \theta(X)\} + \eta(A, X) - \theta(X) + \theta(X) - \psi_{a_0}(P). \qquad \text{(front-door)}$$

Genealogical Relations in Acyclic Directed Mixed Graphs (ADMGs)

- $\tau : X \to A \to M \to Y$ Topological ordering of variables in Figure 1(c).
- $\text{pa}_{\mathcal{G}}(O_i) = \{O_j \in O \mid O_j \to O_i\}$ Parents of O_i.
- $\text{pa}_{\mathcal{G}}(O_i) = \{O_j \in O \mid O_i \to O_j\}$ Children of O_i.
- $\text{dis}_{\mathcal{G}}(O_i) = \{O_j \in O \mid O_j \leftrightarrow \ldots \leftrightarrow O_i\}$. $O_i \in \text{dis}_{\mathcal{G}}(O_i)$ District of O_i.
- $\text{mp}_{\mathcal{G}}(O_i) = \{O_j \prec_\tau O_i \mid O_j \in \text{dis}_{\mathcal{G}}(O_i) \cup \text{pa}_{\mathcal{G}}(\text{dis}_{\mathcal{G}}(O_i))\}$ Markov pillow of O_i.

Identification via Primal Fixability Criterion

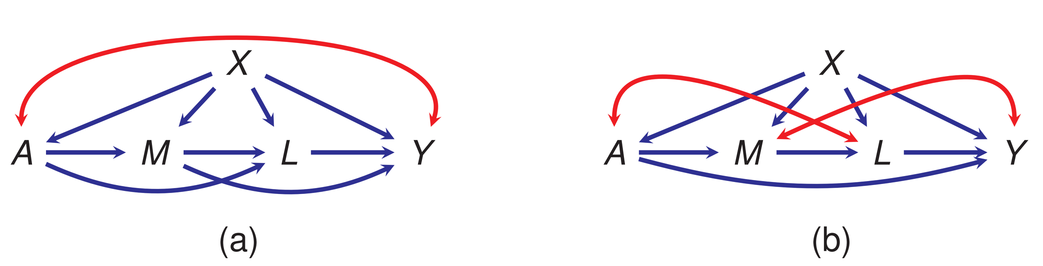


Figure 2: (a) A and Y share the same district; (b) A and Y belong to different districts.

In an ADMG $\mathcal{G}(O)$, $O_i \in O$ is **primal fixable** if $\text{ch}_{\mathcal{G}}(O_i) \cap \text{dis}_{\mathcal{G}}(O_i) = \emptyset$.

- Treatment primal fixability $\Leftrightarrow$ the causal effect of A on $O \backslash A$ is identified[2].

$$\psi_{a_0}(P) = \mathbb{E}\left[p(a_1 \mid \text{mp}_{\mathcal{G}}(A)) \int y\ dP(y \mid \text{mp}_{\mathcal{G}}^{-a}(y), a_Y) \prod_{Z_k \in \mathscr{Z}} dP(z_k \mid \text{mp}_{\mathcal{G}}^{-a}(z_k), a_{Z_k})\right] + \mathbb{E}\left[\mathbb{I}(A = a_0) Y\right] .$$

$(\tau : A \to Z_1 \to \cdots \to Z_k \to Y; \qquad a_{O_i} = a_1$ if $O_i \in \text{dis}_{\mathcal{G}}(A)$ & a_0 otherwise.)

Estimation

Plug-in Estimator

$$\psi_{a_0}(P) = \mathbb{E}\Bigg\{\underbrace{\mathbb{E}\Big[\cdots \underbrace{\mathbb{E}\left[\mathbb{E}(Y \mid \text{mp}_{\mathcal{G}}^{-A}(Y), a_Y) \mid \text{mp}_{\mathcal{G}}^{-A}(Z_k), a_{Z_k}\right]}_{\mathscr{B}_{Z_k}} \cdots \mid \text{mp}_{\mathcal{G}}^{-A}(Z_1), a_{Z_1}\Big]}_{\mathscr{B}_{Z_1}}\Bigg\} + \mathbb{E}\left(\mathbb{I}(A = a_0) Y\right)$$

Example w/ front-door model $\tau : X \to A \to Z_1(M) \to Y$

1. $\hat{\mathbb{E}}(Y \mid \text{mp}_{\mathcal{G}}^{-A}(Y), a_Y)$: $Y \sim M + A + X$, prediction under $A = a_Y = a_1$
2. $\hat{\mathscr{B}}_{Z_1}(X)$: $\hat{\mathbb{E}}(Y \mid Y, a_Y) \sim A + X$, prediction under $A = a_{Z_1} = a_0$
3. $\hat{\mathbb{E}}\{\hat{\mathscr{B}}_{Z_1}\} = \frac{1}{n}\sum_{i=1}^{n} \hat{\mathscr{B}}_{Z_1}(X_i)$
4. $\mathbb{E}(\mathbb{I}(A = a_0) Y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(A_i = a_0) Y_i$

- First-order bias:
$$\psi_{a_0}(\hat{P}) = \psi(P) - \underbrace{P\Phi(\hat{P})}_{\text{first-order bias}} + \underbrace{R_2(\hat{P}, P)}_{\text{remainder term}} \quad \text{(von Mises Expansion).}$$
- Efficient influence function $\Phi(P)$[1]

$$\begin{aligned}\Phi_{a_0}(P) = {} & \mathbb{I}(A = a_Y)\, \mathscr{R}_Y(\text{mp}_{\mathcal{G},\prec}^{-A}(Y)) \left\{Y - \mu(\text{mp}_{\mathcal{G}}^{-A}(Y), a_Y)\right\} \\ & + \sum_{Z_k \in \mathscr{Z}} \mathbb{I}(A = a_{Z_k})\, \mathscr{R}_{Z_k}(\text{mp}_{\mathcal{G},\prec}^{-A}(Z_k)) \\ & \quad \times \left\{\mathscr{B}_{Z_{k+1}}(\text{mp}_{\mathcal{G}}^{-A}(Z_{k+1}), a_{Z_{k+1}}) - \mathscr{B}_{Z_k}(\text{mp}_{\mathcal{G}}^{-A}(Z_k), a_{Z_k})\right\} \\ & + \{\mathbb{I}(A = a_1) - \pi(a_1 \mid \text{mp}_{\mathcal{G}}(A))\}\ \mathscr{B}_{Z_1}(\text{mp}_{\mathcal{G}}^{-A}(Z_1), a_0) \\ & + \pi(a_1 \mid \text{mp}_{\mathcal{G}}(A))\ \mathscr{B}_{Z_1}(\text{mp}_{\mathcal{G}}^{-A}(Z_1), a_0) + \mathbb{I}(A = a_0) Y - \psi_{a_0}(P) ,\end{aligned}$$

where

$$f_{Z_k}^r(Z_k, \text{mp}_{\mathcal{G}}^{-A}(Z_k)) = \frac{f_{Z_k}(Z_k \mid \text{mp}_{\mathcal{G}}^{-A}(Z_k), a_{Z_k})}{f_{Z_k}(Z_k \mid \text{mp}_{\mathcal{G}}^{-A}(Z_k), 1 - a_{Z_k})} , \quad f_A^r(\text{mp}_{\mathcal{G}}(A)) = \frac{\pi(a_1 \mid \text{mp}_{\mathcal{G}}(A))}{\pi(a_0 \mid \text{mp}_{\mathcal{G}}(A))}$$

$$\mathscr{R}_{Z_k}(\text{mp}_{\mathcal{G},\prec}^{-A}(Z_k)) = \begin{cases}\prod_{M_i \in \mathcal{M}_{\prec Z_k}} f_{M_i}^r(M_i, \text{mp}_{\mathcal{G}}^{-A}(M_i)) & Z_k \in \text{dis}_{\mathcal{G}}(A) \\ f_A^r(\text{mp}_{\mathcal{G}}(A)) \prod_{L_i \in \mathcal{L}_{\prec Z_k} \backslash A} f_{L_i}^r(L_i, \text{mp}_{\mathcal{G}}^{-A}(L_i)) & Z_k \notin \text{dis}_{\mathcal{G}}(A).\end{cases}$$

EIF based One-step estimator

- Estimate $\{\mu, \mathscr{B}_{Z_1}, \cdots \mathscr{B}_{Z_k}\}$ via sequential regression
- Estimate density ratio directly or via Bayes' rule:
$$f_{Z_1}^r(m, a, x) = \frac{p(M \mid a_0, X)}{p(M \mid a_1, X)} = \frac{p(a_0 \mid M, X)}{p(a_1 \mid M, X)} \times \frac{p(a_1 \mid X)}{p(a_0 \mid X)}.$$
- **One-step estimator**:
$$\psi_{a_0}^{+}(\hat{Q}) = \frac{1}{n}\sum_{i=1}^{n} g(O_i),$$

where $g(O)$ denotes the functional derived by excluding $-\psi_{a_0}(P)$ in $\Phi_{a_0}(P)$.

Targeted Minimum Loss Based estimator (TMLE)

- Get initial nuisance estimates $\hat{Q} = \{\hat{\mu}, \hat{\mathscr{R}}_Y, \{\hat{\mathscr{R}}_{Z_k}, \hat{\mathscr{B}}_{Z_k} \forall Z_k \in \mathscr{Z}\}\}$
- Update $\hat{Q}$ to $\hat{Q}^*$ via targeting procedure, ensuring that $P_n\Phi(\hat{Q}^*) = o_P(n^{-1/2})$
- **TMLE**
$$\psi_{a_0}(\hat{Q}^*) = \frac{1}{n}\sum_{j=1}^{n}\left\{\hat{\pi}^*(a_1 \mid \text{mp}_{\mathcal{G}}(A_j))\ \hat{\mathscr{B}}_{Z_1}^*(\text{mp}_{\mathcal{G}}^{-A}(Z_{1j}), a_{Z_1}) + \mathbb{I}(A_j = a_0) Y_j\right\} .$$

Asymptotic Properties & Robustness Behaviors

Second-Order Remainder Term

For any $\tilde{Q} = \{\tilde{\mu}, \tilde{\mathscr{R}}_Y, \{\tilde{\mathscr{R}}_{Z_k}, \tilde{\mathscr{B}}_{Z_k} \forall Z_k \in \mathscr{Z}\}\}$, we have:

$$\begin{aligned}R_2(\tilde{Q}, Q) = {} & \sum_{Z_k \in \mathscr{Z}} \int \{\tilde{\mathscr{R}}_{Z_k} - \mathscr{R}_{Z_k}\}(\text{mp}_{\mathcal{G},\prec}^{-a}(z_k)) \times \{\mathscr{B}_{Z_k} - \tilde{\mathscr{B}}_{Z_k}\}(\text{mp}_{\mathcal{G}}^{-a}(z_k), a_{Z_k})\ dP(\text{mp}_{\mathcal{G}}^{-a}(z_k), a_{Z_k}) \\ & + \int \{\tilde{\mathscr{R}}_Y - \mathscr{R}_Y\}(\text{mp}_{\mathcal{G},\prec}^{-a}(y)) \times \{\mu - \tilde{\mu}\}(\text{mp}_{\mathcal{G}}^{-a}(y), a_Y)\ dP(\text{mp}_{\mathcal{G}}^{-a}(y), a_Y) \\ & + \frac{1}{n}\sum_{i=1}^{n}[\mathbb{I}(A_i = a_0) Y_i] - \mathbb{E}[\mathbb{I}(A = a_0) Y] .\end{aligned}$$

Asymptotic linearity

Assume $||\hat{\mu} - \mu|| = (n^{-1/b_Y})$, $||\hat{f}_A^r - f_A^r|| = o_P(n^{-1/r_A})$, and $||\hat{f}_{Z_k}^r - f_{Z_k}^r|| = o_P(n^{-1/r_{Z_k}})$, $||\hat{\mathscr{B}}_{Z_k} - \mathscr{B}_{Z_k}|| = o_P(n^{-1/b_{Z_k}})$, for all $Z_k \in \mathscr{Z}$.

The one-step estimator and TMLE are both asymptotically linear if

1. $\frac{1}{r_{Z_i}} + \frac{1}{b_{Z_k}} \geq \frac{1}{2}$, $\forall Z_k \in \mathscr{Z}$ and $\forall Z_i \in \mathscr{Z}_{\prec Z_k}$ s.t. $a_{Z_i} \neq a_{Z_k}$,
2. $\frac{1}{r_{Z_i}} + \frac{1}{b_Y} \geq \frac{1}{2}$, $\forall Z_i \in \mathscr{Z}_{\prec Y}$ s.t. $a_{Z_i} \neq a_Y$,
3. $\frac{1}{r_A} + \frac{1}{b_{Z_k}} \geq \frac{1}{2}$, $\forall Z_k \in \mathcal{M}$.

Consistency

The one-step estimator and TMLE are consistent estimators for ψ_{a_0} if

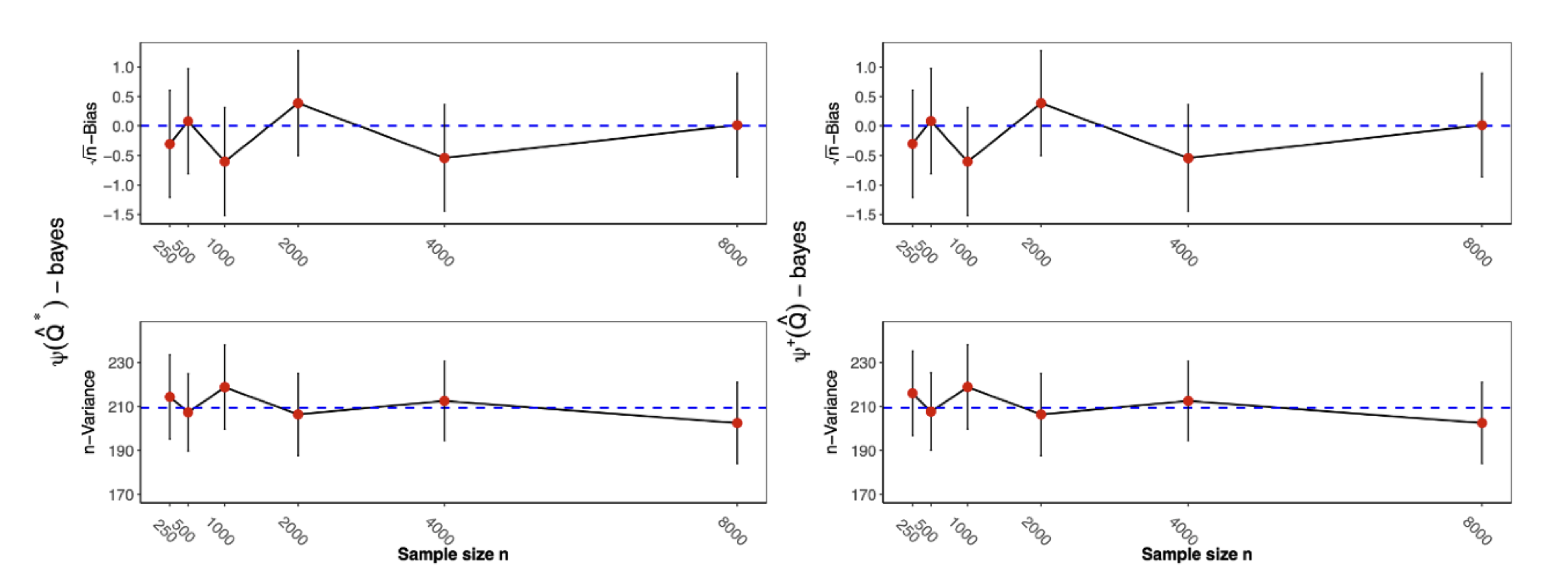
1. $||\hat{\mathscr{B}}_{Z_k} - \mathscr{B}_{Z_k}|| = o_P(1)$ or $||\hat{f}_{Z_i}^r - f_{Z_i}^r|| = o_P(1)$, $\forall Z_k \in \mathscr{Z}$ and $\forall Z_i \in \mathscr{Z}_{\prec Z_k}$ s.t. $a_{Z_i} \neq a_{Z_k}$,
2. $||\hat{\mu} - \mu|| = o_P(1)$ or $||\hat{f}_{Z_i}^r - f_{Z_i}^r|| = o_P(1)$, $\forall Z_i \in \mathscr{Z}$ s.t. $a_{Z_i} \neq a_Y$,
3. $||\hat{\mathscr{B}}_{Z_k} - \mathscr{B}_{Z_k}|| = o_P(1)$ or $||\hat{f}_A^r - f_A^r|| = o_P(1)$, $\forall Z_k \in \mathcal{M}$.

Example w/ front-door model: The estimators are consistent if (i) $(\mu, \mathscr{B}_{Z_1})$ or (ii) $(f_{Z_1}^r, f_A^r)$ is consistently estimated $\Longrightarrow$ **Doubly robust estimators.**

Implementation via `flexCausal` package in R

```
> library(flexCausal)
> est <- ADMGtmle(a=c(1,0),data=data_fig_4a, vertices=c('A','M','L','Y','X'),
+                 bi_edges=list(c('A','Y')),
+                 di_edges=list(c('X','A'), c('X','M'), c('X','L'),c('X','Y'), c('M','Y'), c('A','M'),
+                               c('A','L'), c('M','L'), c('L','Y')),
+                 treatment='A', outcome='Y',
+                 multivariate.variables = list(M=c('M.1','M.2')))
The treatment is not fixable but is primal fixable. Estimation provided via extended front-door adjustment.
TMLE estimated ACE: 1.94; 95% CI: (1.31, 2.57)
Onestep estimated ACE: 1.94; 95% CI: (1.32, 2.56)
 The graph is nonparametrically saturated. Results from the one-step estimator and TMLE are provided, which
are in theory the most efficient estimators.
```

Simulation Studies: Example w/ Figure 2(a)



References

[1] Rohit Bhattacharya, Razieh Nabi, and Ilya Shpitser. Semiparametric inference for causal effects in graphical models with hidden variables. *Journal of Machine Learning Research*, 23(295):1–76, 2022.

[2] Jin Tian and Judea Pearl. A general identification condition for causal effects. In *Eighteenth National Conference on Artificial Intelligence*, pages 567–573, 2002.